THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

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OUR LOCAL BRANCH.

THE second meeting of the North Wales branch was held at the Friars School, Bangor, the headmaster extending a cordial welcome to the members present. In a discussion on the teaching of arithmetic in elementary schools, a local elementary teacher urged the substitution of experimental work in measuring and the practical use of weights and measures for the usual abstract methods, in the case of children as young as 7 or 8, and the president (Prof. Bryan) recommended the use of slide rules at an early stage, and deprecated the working of decimals beyond one or two places. Dr. Bryan also urged that algebra should disappear as a separate examination subject in all schools, pointing out that as the old "book-work and rider" questions in Euclid had been replaced by geometrical examples to be worked with ruler and compasses, so practical work (such as graphs) was wanted in algebra, united with arithmetic, and equations should be reached much earlier by omitting long division, complex fractions, etc. In the discussion that followed it was thought that the change ought to be gradual, so as to avoid confusion in young minds.

"SHOULD SECONDARY TEACHERS BE CIVIL SERVANTS?"

A PAPER by Prof. M. E. Sadler, read before the General Meeting of the Assistant Masters' Association, on January 10th, 1908. The full report appeared in the February number of *The A.M.A.*, the Journal of the Association. Copies can be obtained on application to the Secretary of the Association, 27 Great James Street, London, W.C., price 3d. each, including postage.

THE MATHEMATICAL ASSOCIATION.

REPORT OF THE COUNCIL FOR 1907.

THE Council has to deplore the removal by death of three of the members of the Association during the past year. Mr. E. Budden was a useful member whose name will be familiar to readers of the *Gazette*, to which he was a regular contributor of notes and reviews. He was ever ready to give up part of the leisure of a busy life to work for the *Gazette*; indeed, at the time of his painfully sudden and unexpected death he had reviews for the *Gazette* in hand. Mr. J. C. Palmer was for some years a helpful member of the Council, and attended its meetings and the general meetings of the Association with unfailing regularity. It is many years since the late Lord Kelvin conferred distinction upon the Association by accepting its honorary membership. Although he took no active part in the special work of the Association, yet its endeavours to promote better methods of teaching elementary mathematics had his silent sympathy and approval.

Since the last annual meeting thirty-eight new members have been elected, and the total number of members at the present time is 433. Of these 9 are honorary members, 25 are life members by composition, and from 352 an annual subscription is expected. The remainder, 47, are exempt from payment of the annual subscription under the rule which has now been rescinded.

A suggestion having been made to the Council that it would meet the convenience of many members of the Association if the annual meeting were held earlier in January, before the schools reassembled, a voting card was sent out, so that each member might have an opportunity of expressing his wishes. The alternative dates offered were Friday, January 8th, and Saturday, January 25th. The number of replies received was 177. Of these 65 were in favour of the earlier date, and 60 in favour of the later; the remainder were practically neutral. The Council came to the conclusion that the preponderance in favour of the earlier date was not sufficient to justify them in making any change this year.

Early in the year the Council considered the question of obtaining a grant of incorporation, but eventually they decided that the advantages to be gained by incorporation would not justify them in recommending the Association to incur the necessary expenditure, which was estimated at about £55.

The Council have also had under consideration the revision of the rules, which have not been amended since 1898. The time appeared suitable also, because the stock of copies was all but exhausted, and reprinting in some form had become necessary.

The matter was referred to a small committee, and the Association will at this meeting have an opportunity of considering the changes which members of that committee will propose.

Considerable difficulty has been experienced from time to time in keeping the addresses of the life members of the Association correct, as the customary communications on financial matters do not in their case pass between them and the Treasurer. The Council have decided therefore that a printed form shall be sent to all such members early in January in each year, and that each of these members shall be requested to return the form immediately to the Honorary Secretaries stating on it the address to which the *Gazette* and all other communications should be sent during the succeeding 12 months. The intention is that no *Gazette* or other communication shall be sent to a life member until the reply has been received.

The report on the Teaching of Mathematics in Preparatory Schools was published in the December *Gazette*, and is also published separately by Messrs. G. Bell & Sons.

The members of the Council who retire at this meeting are Mr. R. F. Davis and Mr. F. W. Dobbs, and nominations of members to fill the two vacancies thus created were asked for in the October *Gazette*. No nominations having been received before the end of November, the Council decided to propose that Mr. David Mair of the Civil Service Commission, and Mr. C. C. Lynam of the Oxford Preparatory School, be elected to fill the vacancies. The Council propose also that Mr. G. B. Mathews, F.R.S. (President 1904 and 1905), be elected a Vice-President, and that Mr. R. F. Davis be re-elected Auditor.

The Council desire to express their cordial appreciation, and that of every member of the Association, of the ability and hard work which Mr. Greenstreet continues to place at their service as Editor of the *Mathematical Gazette*.

The Council desire also to express their thanks to Mr. C. S. Jackson for the excellent arrangements which he made on the occasion of the third annual dinner last January; and to the Council of King's College for granting to the Association facilities for its meetings.

Report of the Committee of the Mathematical Association on the Course of Mathematics required for Entrance Scholarships at the Universities.

1. It is very desirable that the subjects of examination for Mathematical Entrance Scholarships at Cambridge and Oxford should admit of the same course of preparation.

2. In order to allow of a broader syllabus the exercises and riders should be simplified.

3. Questions on elementary bookwork ought not to be entirely omitted.

Restricting bookwork questions to the higher bookwork would encourage the neglect of the more elementary parts, which are of at least equal importance.

The Committee considers that the essay type of bookwork questions (as set in the Trinity Group of Colleges at Cambridge in December, 1905) is of special value.

4. A knowledge of the general principles and application of graphical work should be required, but only a moderate degree of manipulative dexterity.

5. The range in Pure Geometry should approximate to the Oxford standard: purely geometrical methods should be encouraged by a fair number of questions being set which can best be solved by pure Geometry. Purely geometrical methods as opposed to analytical should not be insisted upon, but there is no objection to the appropriate method of solution being recommended.

6. In the Calculus, geometrical proofs, which assume that the functions considered are such as can be represented by curves, should be permitted, provided that the necessary conditions for their validity are stated.

7. In Geometry, Trigonometry, Theory of Equations, and Dynamics the use of Infinitesimals and the methods of the Calculus should be freely permitted.

8. The subjects of the examination should be in accordance with the Schedule appended, due regard being paid to the second recommendation of the Committee.

SCHEDULE.

PURE GEOMETRY.

Geometry of Straight Lines, Circles and Conics; Inversion, Cross-ratios, Involution, Homographic Ranges, Projection, Reciprocation and Principle of Duality; Elementary Solid Geometry, including Plans and Elevations of Simple Solids.

ANALYTICAL GEOMETRY.

Straight Lines and Curves of the Second Degree: Tangential Coordinates.

Excluding Homogeneous Coordinates; Invariants; and Analytical Solid Geometry.

ALGEBRA, including Elementary Theory of Equations.

Excluding Recurring Continued Fractions; harder tests of Convergence; Theory of Numbers; and Probability.

GEOMETRICAL TRIGONOMETRY. *Excluding* Spherical Trigonometry.

ANALYTICAL TRIGONOMETRY.

Properties of Circular, Hyperbolic, Exponential and Logarithmic Functions, with real and complex argument.

Excluding the *proofs* of the Infinite Products for Sine and Cosine, and of the Series of Partial Fractions for the other Trigonometrical Ratios.

CALCULUS.

Total and Partial Differentiation; Taylor's and Maclaurin's Theorems; Elementary Integral Calculus; Simple Applications to Plane Curves (especially to such as are of intrinsic importance), to Maxima and Minima, to Areas and Volumes, and to Dynamics; Curve Tracing, not as a rule to scale.

Excluding the Theory (but not the use) of Differential Equations.

DYNAMICS.

Elementary Statics, including Simple Graphical Statics; Elementary Kinematics and Kinetics, including motion of a rigid body about a fixed axis, and motion of cylinders and spheres in cases where the centre of gravity describes a straight line.

Excluding Hydrostatics and Hydrodynamics.

THE PRESIDENTIAL ADDRESS, 1908.

"THE USES OF MATHEMATICS AND THE TRAINING OF THE MATHEMATICAL TEACHER."

WHAT is the use of mathematics?

There are many parents who would find it extremely difficult to answer this question, and if they attempted to give an answer, they would in many cases repeat what appears to be a widespread current belief, namely, that mathematics is of no use in itself, but that a school course in the subject affords valuable mental training and develops the brain.

If this were all that could be said in favour of mathematics, a strong case could be made out for the rival claims of *philately* to be included in the curricula of our schools and universities. The intelligent study of a collection of postage stamps may be made of greater educational value than the unintelligent manipulation of mere meaningless collections of algebraic symbols. The detection of watermarks and the measurement of perforations may be used to afford valuable mental training, and are well calculated to develop a child's powers of observation and accurate measurement.

This statement of the educational claims of philately must not be taken seriously. It must only be regarded as indicating that, if mathematics is to be defended against the many competing claims for recognition of other subjects in our educational systems, a much stronger case must be made out than that based on the mere academic ideal and the considerations associated with it as to the value of the mental training afforded by the study of mathematics.

Now there are two aspects of mathematics which in my opinion place that subject far and away in front of any other branch of human study. These are the research aspect and the practical aspect. Of both of these the British public is as a rule profoundly ignorant. The claims of scientific research for recognition are being slowly forced upon our country by the successes of Germany and America, but those who associate any meaning with the word "research," contemplate the building of palatial laboratories to which the mathematician finds no admittance. Yet the mathematical researches of to-day become the practical applications of to-morrow and culminate in the commercial successes of the near future.

It cannot be too strongly made known that the mathematician who shuts himself up in his study, and who is content to live on a starvation income turning out pages of foolscap covered with x's and y's, may be a pioneer, who, without knowing it, is building up the foundations of a great industry.

But, people will say, this may be very true of applied mathematics, but surely those long and mystical hieroglyphics in pure mathematics can never lead to anything practical.

A greater mistake could not be made.

When the engineer passes on a problem to the applied mathematician, it is the want of further knowledge of the highest and most advanced parts of pure mathematics which frequently stands in the way of the final solution.

Let us take one example of the way in which the study of what appeared to be purely symbolical abstractions has conduced to the progress of civilisation.

When mathematicians first began to solve quadratic equations, they soon found the square roots of negative quantities occurring in their solutions, and a "practical man" of the ordinary type would naturally infer that the problems could not be solved, and that it would not pay

to study them any further. It used to be narrated at Cambridge that Todhunter once dreamt that he was minus one under a root and could not extract himself. This story was in all probability an invention. But the further study of imaginary quantities has revealed the fact that this square root of minus one is a magician's wand which is capable of transforming a subsiding disturbance into an oscillatory one; which is capable of tracing out electrical lines of force, and thus calculating the capacity of condensers, which with equal facility can be utilised to plot the stream lines in fluid motion and thus solve problems of great value in connection with naval architecture, and which has played an indispensable part in the development both of ordinary and of wireless telegraphy and in the transmission and utilisation of electric power. In lecturing on complex quantities to a class containing several electrical engineering students, I am to point out to them that the subject of the lectures is *engineering* quite as much as mathematics.

In stating that the future welfare of mankind lies to a very large extent in the hands of the mathematician, I wish it to be understood that the mathematician is the custodian of a priceless treasure, and that a heavy responsibility rests on his shoulders to use the gifts with which he has been endowed for the benefit of the race.

The knowledge of this truth grows upon me every year, but it was not till many years after I had completed my training for the Cambridge Tripos that the truth even dawned on me. The period following the Tripos was one of depression, arising from the scarcity of openings and lucrative appointments available for mathematicians with First Class degrees, and the large number of "unemployed" who were endeavouring to find precarious work by coaching up others, who in their turn would also join the "ranks of the unemployed."

It was then that I was introduced to the research aspect of mathematics by the stimulating influence of several of the mathematical lecturers of St. John's College, an influence which has led me always to regard St. John's as *my college*, even though the bond of connection is entirely unofficial, and can therefore be described in mathematical language as "imaginary." It is quite as strong a connection as that which exists between a circle and a straight line outside it which intersect in two imaginary points.

But the researches were for a long time regarded as mere additions to the world's ever-accumulating hoard of printed literature which might or might not be read by a small circle of mutually admiring mathematicians, and which might be regarded as buried rubbish unless some other mathematician chanced to like to work out one of the subjects further, in which case his paper would follow the same fate. It was not till several years later that the practical aspect of mathematics dawned on me. I had read a paper on "Buckling of Plates" at the British Association, and this brought me into contact with the late Mr. G. R. Durrell, for many years editor of *Engineering*, who introduced me to the discussions at the annual meetings of the Institution of Naval Architects. In these discussions I found mathematical considerations cropping up everywhere, and what struck me most was that the problems which the naval architect has to deal with involve not elementary mathematics, but the most advanced knowledge which the mathematician can bring to bear on them. At the present time there is abundance of material for mathematical research in the problems of naval architecture presented by the papers read at these interesting gatherings, and the Institution is only too glad to receive papers on them.

A different spirit exists unfortunately among many practical men who regard the mathematician as a clever but incomprehensible being whose work is of no practical value, and who in fact frequently arrives at

results which will not hold the test of experimental verification. Such men quote the story of the Cambridge mathematician who calculated the height of a steeple at eight inches. Now when such discrepancies occur the usual fault is that the so-called mathematician is *not sufficient of a mathematician*. It is his want of *more* mathematical knowledge that causes him to arrive at a false conclusion. Such occurrences are calculated to give the public a very false idea of the value of mathematics, and further harm is done by the effusions of the many circle squarers, angle trisectors and cube duplicators who pose as mathematicians before those who know no better.

A further difficulty arises from the vastness of the subject. No man can ever understand more than a small fraction of our existing mathematical literature. The solution of a problem in naval architecture may require the brains of several mathematicians to be brought to bear on it, and mathematicians in this country are so overworked that the difficulties may be likened to those which I have experienced in obtaining the necessary instrumental parts for a performance of Schumann's Quintet with piano-player accompaniment.

The position of mathematicians in Great Britain is infinitely worse than it is in Germany, France or America. It is essential for our national progress that mathematicians should bestir themselves, should bring their claims prominently before the public, should establish their proper position as useful members of the community, and should fight against the prevalent spirit of indifference and apathy with which their work is commonly regarded, a spirit for which they themselves are often largely to blame.

People cannot be expected to hunt about under bushels on the off-chance of finding a candle. The late Lord Kelvin was no cupboard philosopher. When a new electrical or semi-scientific journal was started his portrait frequently appeared on the frontispiece. This became amusing at times, but it was the right thing.

But our attempts to establish our proper position in the world must begin in the schools. It is no use expecting people of mature years, every hour of whose time is fully occupied, to readily change their views regarding the utility of mathematics. The younger generation must be taught to regard mathematics as a science and as something useful, not as an examination subject.

The important work which has been accomplished in recent years by our Association, to a great measure, I believe, as the result of a movement for which John Perry is largely responsible, has greatly improved our position. In the years before our Committee had drawn up its Report, a number of Euclids had been placed on the market. Immediately the Report was published came the rush to Klondyke. A flood of School Geometries was poured out, and the Euclids were written off as losses. The boys who used to learn the bookwork and could never do a rider now have plenty of occupation for their brains and fingers with ruler and compasses. They are able to do something and to find out something.

But there has been no corresponding flood of Algebras, and the old addition, subtraction, multiplication, division, greatest common measure, least common multiple, square root and fractions die hard. The immediate result of the change was in most cases a "chapter on graphs," which fitted in ill with the context. Thus the movement which has for its object the lightening of the work of both teacher and pupil in the elementary stages, has in many cases had the contrary effect, and teachers are complaining that in addition to their old duties they have to teach geometrical drawing and graphs. During the last few years a few Algebras on improved lines have been announced. But the schoolboy will never overcome his dislike for mathematics as long as he is required to perform examination drill with

mere collections of symbols. The logical solution of the difficulty is that Algebra should either absorb or be almost completely absorbed by other subjects, and that the manipulation of algebraical formulae should never be taught except in connection with applications to practical computation. Personally I consider that the time has arrived for the disappearance of algebra as a separate examination subject. No paper should be headed with the forbidding title "Algebra." The division of mathematics into compartments separated by watertight bulkheads has been condemned by many competent authorities. The result of this method of doling out algebra and geometry in separate dishes is that many schoolboys think that "the answer that is asked for in the book" is something different from what they would obtain if they used their own common sense. They would calculate the price of so many pounds of sugar quite correctly if they were in a grocer's shop, but when they sit over their books they give an answer bringing in $\frac{121}{256}$ of a penny, or running into millions of pounds.

As the recommendations of the M.A. Committee have filtered through the teaching world their diffusion has been retarded and their potency diluted by the mass of old-fangled ideas pervading that world. It says a great deal for the success of the movement that boys when they leave school are now able to perform much more intelligent work in geometry than they did a few years ago.

But if the teaching of mathematics is to be put on a really sound footing the teachers themselves must be trained to an appreciation of the research ideal in mathematics. A teacher of physics or chemistry usually knows something about research, why should a teacher of mathematics be ignorant of the work that is going on in the world of science? It is this ignorance that is largely responsible for England's neglect of mathematics.

What is wanted is that *every university student who intends to qualify for teaching mathematics should attend one course of lectures on one particular branch of higher mathematics, given by a professor who is an authority on that special subject and who has carried out researches in it.*

This, I believe, sounds rather a large order, but it is quite capable of realisation. We in Wales are working towards this end, and have made very substantial progress in the desired direction. We are hampered by many difficulties which do not exist at the older universities. The educational system of Wales is of comparatively recent development. Moreover, our students are not able to devote the whole of their period of academic study to mathematics alone. They must also take a full pass degree course in physics or chemistry or a number of courses in literary subjects or branches of science. Yet every candidate for mathematical honours is required to specialise in some one branch of higher mathematics and to pass an examination in this subject. A recent important advance has been the adoption of the open book in this examination; moreover, the candidate's lecture notes are used by him in the examination, and at the conclusion are sent up to the Examining Board. We also require a dissertation to be submitted as a qualification for the degree of *Magister*. When I further state that Welsh students, despite the heaviness of their courses, have satisfactorily stood the tests on such advanced subjects as Grassmann's *Ausdehnungslehre*, and have actually sent up dissertations for the M.A. degree, one student of mine choosing a subject entirely different to anything taught in my department, it will be seen that my proposal is perfectly practicable.

I do not say that every university need necessarily require an examination on the selected subject. But a certificate of attendance on a course of higher mathematics should form one of the necessary qualifications for admission to an honours examination, and the certificate should include a statement that the candidate's lecture notes had been read and approved.

By an arrangement such as I have suggested the dead level of uniformity in the training of our teachers would disappear, and the hard and fast

barrier which so often separates the mathematician from the mathematical teacher would break down. We should have all sorts and conditions of men and women teaching mathematics in our schools, and the variety in their university training would produce representatives acquainted with every branch of mathematics and capable of reading and appreciating papers (not necessarily of doing actual research) on the prominent problems of the day. Thus would be forged a link in the chain required to connect the mathematician with the British practical man.

The older universities have possessed exceptional opportunities for carrying out such a reform. The subdivision of the tripos at Cambridge into two parts might have been adopted as the basis, the second part allowing of the necessary specialisation after a general training had been provided in Part I. But from the first Part II. was doomed to be a failure, partly because the Senior Wranglership was associated with Part I., and partly because in the first stages none but wranglers were admitted to Part II. What was wanted was to make Part I. and Part II. both integral parts of the training of the average mathematical teacher and to popularise the Second and Third Classes in Part II. The new regulations might easily have accomplished this. But they will probably have the reverse effect, partly because the selected subject is only to be regarded as a qualification for distinction, and partly because Cambridge has killed the goose which laid the golden eggs in the form of the Senior Wrangler. Now, the Senior Wrangler has been a fetish which appealed greatly to the dailymailocratic instincts of the British public. There was no need whatever to kill him. The proper course was to assign this title of honour to anyone who obtained sufficiently high distinction in Part II. In a good year there might be several Senior Wranglers, in a poor year none. The Senior Wranglership should, in fact, have taken the place of the "One One" in Part II. Those who failed to obtain it would still have been awarded the honourable title of wrangler instead of the old position of Class I. Division 2, which implies failure to attain something better. But Cambridge gave no one a chance of voting except on a choice between two evils. The present evil is that the schools will appoint their teachers on the Part I. degree alone.

There are plenty of young mathematicians at Cambridge who would only be too glad to discharge the duties of *Privatdozent*, for which they are eminently qualified. But it was prominently stated in the fly-sheets that the University professors, lecturers and Privatdozenten at Cambridge *never come into touch with the students*, and that therefore little value should be attached to their opinions on the position of mathematics in the University. Fancy such an allegation being put forward seriously in the twentieth century! What can Germany and America think of our leading Universities?

A knowledge of the research ideal is necessary to develop originality on the part of the teacher. There is more need of originality now than there ever has been in the past, and this need is most acutely felt in teaching the elementary parts of mathematics. It is only by the exercise of originality that the recommendations of the M.A. can bring forth the best fruit. Outside geometry we want original text books on nearly every part of the school curriculum. But the writing of text books is in most cases a thankless task, and it is very doubtful how far America may be able to supply our needs, so that a great responsibility devolves on our teachers.

In addition, the teacher requires some general kind of knowledge about certain important branches of current research. No man would be considered properly qualified to teach physics in a school who was not at least aware of the existence of Röntgen rays, radium and wireless telegraphy. The mathematical master should at any rate be equipped with a similar small modicum of information regarding such modern fields of discovery as the Theory of Groups and Non-Euclidian Geometry. As he will be largely

occupied in teaching geometry, his knowledge of the latter subject should extend at least to the contents of one of the little pamphlets published in France on "our conceptions of space" which are intended for non-mathematical readers. Otherwise he cannot form a proper appreciation of the geometry which he is teaching, and he may lay undue stress on points which could with advantage be omitted. Yet I have known the case of a high wrangler who, on being asked whether he had given any attention to non-Euclidian geometry, referred the enquirer to a copy of Baker and Bourne!

If the above remarks refer to the ideal to be aimed at in the training of the secondary teacher, what shall we say about the elementary teacher? His case is about as bad as it can possibly be. It is not to be wondered at that the attempts of our Association to enlist the elementary teacher in our ranks have met with such a scanty response. The fault lies in the way he (or she) has been trained. He never regards mathematics as anything but an examination subject, and his examinations are frequently conducted on very antiquated and unpractical lines. He may even be required to closely follow Euclid's treatment in order to pass in geometry. I happened one day some years ago to glance over the exercises of one of these students, and I found that she was adopting a very long and laborious method to work out results that could be done in a line by better and simpler methods, which I offered to show her. "It's really no use," was the answer; "this is the way we have to do it for our examination, and I haven't time to learn any other way."

There is no place for the research ideal or the academic ideal in the training of the elementary teacher. His course should be drawn up exclusively on practical lines. His proper duty is to give the children of the working man such an education as will make them efficient British subjects, and to endow them with such qualities as will enable them to rise to the best positions in the labour market. Unless elementary education is conducted with this object in view the labouring classes have quite as strong a cause for complaint as the taxpayers. Such exercises as simplification of fractions (including the $3\frac{3}{4}$ of $1\frac{7}{8}$ divided by $1\frac{5}{16}$ of our old books) have no proper place in the curriculum of the labourer's son. What he wants is to be able to measure and calculate properly. He might be given pieces of cardboard representing flagstones, and asked to arrange them in the best way to pave a footpath. A heap of sand or gravel might be shot down in the school yard, and he might be asked to find how many tons of sand it contains. He would measure the height and diameter of the heap, and would then take, say, a cubic foot of sand in a biscuit tin and weigh it with a pair of scales. He might plot a piece of land to scale, using a protractor when he was sufficiently advanced, and might measure its area. Then with the protractor he might draw figures and measure heights and distances. He might in mechanics learn the principle of the lever, and learn that you can never get more work out of a machine than you put into it, and that you generally get a great deal less. And he should know enough about hydraulics to understand when anything goes wrong with the water supply, and not to lay drains sloping upwards, as he commonly does.

He should learn to use a slide rule from the very beginning. There is no reason why an addition slide rule should not be one of the first toys that a child has to play with. It will help him to count and to say 2 and 2 are 4, 2 and 3 are 5. A multiplication or ordinary slide rule can pretty well be introduced in learning the multiplication table. Then will disappear the troubles with multiplying decimals, and the resolution of our Association as to not requiring the use of contracted methods will be understood in its true light.

As it is, I greatly fear that this resolution will in too many cases lead to a perpetuation of the long strings of figures which boys will insist in calcu-

lating to a high degree of unmathematical inaccuracy. The mistake is made when boys are taught to calculate products and quotients except in the form of approximations. They then learn something which is not only useless but which does harm, as they cannot unlearn it afterwards. Let them use a slide rule until they can learn to multiply properly.*

Having said that uncontracted results of multiplication are useless, I would go further and say that they are incorrect. The decimal 0·38909 represents the product not of 0·533 and 0·73 but of 0·53300 and 0·73000. An example of this kind of unmathematical inaccuracy appeared in a recent account of Mr. Farman's aeroplane experiments, in which the English equivalents of recorded flights of 771 and 800 metres were given as 2529·52 and 2624·66 feet.

The mathematical training of the elementary teacher is a question of the highest national importance. It must be remodelled in such a way as to give him an insight into the practical uses of mathematics when carried to a higher level than he has to teach. It should certainly include the notation of the Differential and Integral Calculus, together with such practical applications as can be deduced from the differentiation and integration of positive integral powers. He will thus learn that there are a good many problems which require more complicated formulae for their solution, and that what he does for his examination is merely a stepping stone to higher methods on which many of the most important problems of modern civilisation depend.

Unfortunately, however, modern educational authorities seem intent in developing in our labouring classes a spirit of luxury and discontent instead of trying to improve their efficiency and intelligence, and thus the elementary teacher has his time largely taken up with such subjects as music and nature study, of which the latter may be of great use in rural districts, but is out of place in the densely populated towns in which it is introduced. This very fact shows a want of the proper mathematical mind on the part of educational authorities in their failure to take account of the varied conditions of their problem.

Our Association might do very useful work in turning its attention to certain questions of mathematical notation and nomenclature. The unmathematical inaccuracies arising in approximate calculations are largely due to the fact that we have only one symbol 0 to denote "aught" (or as the schoolboys say ought) and nought. When we speak of the Earth's radius as 4000 miles, the 0's stand for aughts, not noughts. It would be impossible to add, subtract, multiply or divide past an aught, and approximate calculations would thus be greatly simplified.

There is also the need of a better inverse notation than that afforded by the index of the first negative power. This notation stands in incongruity with the notation for the sine squared. Despite Todhunter's attempt at justification, it always reminds me of the schoolboy's answer to the question why a certain noun was in the accusative, not the genitive case: "To make it harder for us." It involves the retention of six trigonometric functions when three would otherwise suffice. It is apparently not used for logarithms, for I have never seen $\log^{-1}x=e^x$ or $e^{-1x}=\log x$ written.

In most civilised countries they use arc sin and arc cos. Now the Greek prefix a meets every requirement, and is easier to write than arc. As applied to trigonometric functions a is the short for arc. It is also the short for angle, which is a decided advantage. In dealing with hyperbolic functions a stands for arg or argument; as applied to logarithms, a log may be taken to stand for antilog. The adoption of this simple and suggestive prefix is surely much more useful than the substitution of the "sneeze" and "cough" notation

* The Indian method of long multiplication is in many ways better than the rules commonly taught and does not appear to involve the same difficult process of unlearning when approximate results are required.

for the "shine" and "cosh" with which we are so familiar, a substitution on which certain examining bodies appear to lay special stress.

In conclusion, a vast amount of work lies before our Association waiting to be done. Progress must of necessity be slow, but I hope that in the years to come we shall be successful in our attempts to raise mathematics to a higher level in Great Britain than it has ever occupied in the past. And I again repeat that this much desired end will conduce in no uncertain degree to the welfare and prosperity of our nation.

THE CONIC THROUGH FIVE GIVEN POINTS.

The following discussion may be of interest on account of the way in which it leads to the axial and focal properties.

I. Let the locus of a point P such that $A(CDEP)=B(CDEP)$, where A, B, C, D, E are fixed points, be called the conic (AB, CDE). Then it is clear that permuting A, B *or* C, D, E does not affect the conic, and that the conic passes through C, D, E, and also through A, B. (Give BP the four positions BC, BD, BE, BA and then give AP the position AB.)

Since AP and BP correspond homographically the same conic may be defined as (AB, FGH) where F, G, H are any three positions of P except A, B.

II. If F is any point on (AB, CDE) then Pascal's theorem can be proved for any hexagon whose vertices are A, B, C, D, E, F if B is next but one to A, and thus A or B may be interchanged with any one of C, D, E. For instance if A is to be interchanged with C we put A, B, C 1st, 3rd and 5th as in $ADBECF$. This is a Pascal hexagon, and it follows that F lies on (BC, ADE). Thus the conic is not affected by *any* permutation of the five points after which it is named, and we may call it $(ABCDE)$.

III. If any circle has been drawn through AB, the points P, Q where this circle meets the conic can be constructed with the ruler only as follows.

Let AC, AD, AE, BC, BD, BE meet the circle in $C_1, D_1, E_1, C_2, D_2, E_2$.

Then
$$\begin{aligned} A(C_1D_1E_1P) &= A(CDEP) \\ &= B(CDEP) \\ &= B(C_2D_2E_2P). \end{aligned}$$

Hence P is a double point in the homography in which C_1, D_1, E_1 correspond to C_2, D_2, E_2; the other double point is Q, and thus P, Q are the intersections with the circle of the Pascal line of the hexagon $C_1D_2E_1C_2D_1E_2$.

IV. By III. we may suppose the conic to be named after five points $ABCDE$, of which four are on a circle, say A, B, C, D. Take another circle through AB and let it meet the conic again in P, Q. In the figure of III. we then have
$$\angle D_2C_1A = D_2BA = DBA = DCA.$$
So that C_1D_2 is parallel to CD, as also is C_2D_1. Hence PQ also must be parallel to CD; PQ is therefore in a fixed direction whatever circle through A, B is chosen.

V. As a particular case of IV. suppose C, D both at infinity: then it still follows that
$$D_2C_1A = DBA, \quad D_1C_2B = DAB,$$
and hence C_1D_2, D_1C_2 are parallel and PQ is parallel to them, making the same angle with AC that AD makes with AB.

VI. Still taking $ABCD$ cyclic, suppose that AB meets CD in T, and let the circle, whose centre is T and radius TA, meet CD in K, L. Let G, H be the points where AK, AL meet the conic. (G, H can be found by the

known construction depending on Pascal's theorem.) Construct as in III. the point I, where the circle ABG meets the conic again. Then GI is parallel to CD, and therefore AB, GI are equally inclined to AG and AG, BI are parallel. If any other circle through A, G meets the conic in P, Q, then PQ must be parallel to BI, that is, to AG, and all chords of the conic parallel to AG are therefore bisected by a line perpendicular to them. The same holds for chords parallel to AH, and the conic has therefore two axes of symmetry in general. It follows from V. that G, H cannot both be at infinity, and hence the conic has at least one axis.

VII. If PK, PL, PM, PN are drawn perpendicular to AB, CD, AD, CB, then, as P varies, $P(ABCD)$ is proportional to $PM \cdot PN/PK \cdot PL$, and this is true whether $ABCD$ is cyclic or not.

VIII. If in VII. we suppose that $ABCD$ is cyclic we have that

$$PM \cdot PN + PK \cdot PL$$

$\propto$ the square of the tangent from P to the circle $ABCD$.

Proof. Let PB, PD meet the circle again in E, F, and let EE', FF' be diameters.

Then $\dfrac{PL}{FC} = \dfrac{PM}{FA} = \dfrac{PD}{FF'}$ by similar figures $PLMD$, $FCAF'$,

and $\dfrac{PK}{EA} = \dfrac{PN}{EC} = \dfrac{PB}{EE'}$ by similar figures $PKNB$, $EACE'$.

Also $EA \cdot FC + EC \cdot FA = EF \cdot AC$, and thus

$$\frac{PK \cdot PL + PM \cdot PN}{EF \cdot AC} = \frac{PB \cdot PD}{EE' \cdot FF'}.$$

But $EF/BD = PF/PB$, whence $PK \cdot PL + PM \cdot PN \propto PD \cdot PF$.

Hence if P is on a conic through A, B, C, D, the square of the tangent from P to the circle $ABCD$ is proportional to $PM \cdot PN$ or $PK \cdot PL$.

IX. In VIII. let AB, and therefore also CD, be perpendicular to an axis, which meets AB, CD in U, V, and let G be the foot of the perpendicular from P to this axis. Then from VIII. we have $t^2 = e^2 GU \cdot GV$ where e is a constant and t the tangent from P to the circle $ABCD$. Let O be the centre of this circle and take another circle through A, B with centre O_1; let t_1 be the tangent from P to the new circle. Then

$$t_1^2 - t^2 = 2OO_1 \cdot GU$$

$$\text{and } t_1^2 = GU(e^2 GV + 2OO_1)$$

$$= e^2 GU \cdot GV_1$$

if V_1 is a point on UV such that $e^2 VV_1 = 2OO_1$.

The conic may therefore be defined equally well by reference to the new circle and the lines AB, C_1D_1, where C_1D_1 is the perpendicular to UV through V_1. The points where C_1D_1 meets the new circle lie on the conic. Similarly we may move U forward to U_1 if at the same time we again move the centre of the circle forward to O_2 so that $e^2 \cdot UU_1 = 2O_1O_2$ and now make C_1D_1 the radical axis of the new and old circles. It is not necessary that the new circle should meet the conic in real points.

X. Let r, r_1, r_2 be the radii of the three circles, with centres O, O_1, O_2. We have $r_1^2 - r^2 = UO_1^2 - UO^2$, since the radical axis of the first two passes through U, and similarly $r_2^2 - r_1^2 = V_1O_2^2 - V_1O_1^2$.

Thus $$r_2^2 - r^2 = V_1O_2^2 - V_1O_1^2 + UO_1^2 - UO^2,$$

while $$OO_2 = \tfrac{1}{2}e^2(VV_1 + UU_1) = e^2 \cdot GG_1$$

if G, G_1 are the middle points of UV and U_1V_1.

After some reduction the value of r_2^2, unless $e = 1$, becomes

$$e^2(1 - e^2)XG_1 \cdot G_1X' + \tfrac{1}{4}e^2 \cdot U_1V_1^2$$

where X, X' are two points on the axis such that

$$(1-e^2)(GX+GX')=2GO,$$

$$e^2(1-e^2)GX\,.\,GX'=\tfrac{1}{4}e^2\,.\,UV^2-r^2.$$

Making U_1, V_1, and therefore G_1, to coincide with X or X' we have $r_2=0$; the circle reduces to a point and we have the focus and directrix property.

When $e=1$, the value of r_2^2 becomes $2GO\,.\,XG_1+\frac{1}{4}U_1V_1^2$ where X is a point in the axis such that

$$2GO\,.\,XG=r^2-\tfrac{1}{4}UV^2.$$

Here $r_2=0$ when U_1, V_1, G_1 coincide with X. A. C. Dixon.

MATHEMATICAL NOTES.

253. [1. 2. b.] Cf. Note 249, p. 167.

If n be a prime, a any number prime to n, p the smallest value of the integer m for which a^m-1 is divisible by n and n^s the highest power of n which divides a^p-1, we may write

$$a^p=1+bn^s+c_1n^{s+1}, \quad \text{.....(i)}$$

where $0<b<n$.

Now if P be the smallest value of m for which a^m-1 is divisible by n^{s+1}, P must be a multiple of $p=\lambda p$, say. But from (i),

$$a^{\lambda p}=1+\lambda b\,.\,n^s+\text{a multiple of } n^{s+1}.$$

Therefore λb is divisible by n and (since $b<n$) $\lambda=n$.

Further, since in the expansion of $(A+B+C)^n$ the coefficient of every term except A^n, B^n, or C^n is divisible by n,

$$a^{np}=1+bn^{s+1}+n \text{ (a multiple of } n^{s+1})$$

[unless $n=2$, $s=1$]

$$=1+bn^{s+1}+c_2n^{s+2}.$$

Similarly $\quad a^{n^2p}=1+bn^{s+2}+c_3n^{s+3}$,

and generally $\quad a^{n^qp}=1+bn^{s+q}+c_qn^{s+q}$

for all positive integral values of q.

Thus n^qp is the smallest value of m for which a^m-1 is divisible by n^{s+q}.

The above furnishes an answer to Mr. Wiles's question in Note 249 (*M. G.* Dec. 1907, p. 167). From this we can shew that for a composite number N, if a be a number prime to N, then N^qP is the smallest value of m for which a^m-1 is divisible by N^{s+q}, s, q being integers readily found by an examination of the powers to which the various prime factors of N are raised in $a^{p'}-1$, where p' is the smallest value of m for which a^m-1 is divisible by N.

Since we know that $a^{n-1}-1$ is divisible by n it follows that p is a factor of $n-1$, and the exceptions to the rule as enunciated by Mr. Wiles are determined by finding values of n and a for which $a^{n-1}-1$ is divisible by n^2, n^3, ... n^s.

If a be given, this amounts to finding values of m for which a^m-1 is divisible by x^p, where $x>m$ and $p>1$. This can probably be done only by trial: thus

3^5-1 is divisible by 11^2.

5^1-1 ..... 2^2.

7^4-1 ..... 5^2,

and so on. I do not remember seeing a proof, but I think $2^{n-1}-1$ is never divisible by n^2. Are there other such cases?

If, on the other hand, n be given, there are $n-2$ distinct values (besides 1) of m, for which

$$(m+kn^2)^{n-1}$$

is divisible by n^2 for all integral values of k. Any one of them may be thus determined. Let m_1 be any integer in the limits

$$1 < m_1 < n.$$

Then $$m_1^{n-1} = 1 + \mu n + \mu' n^2,$$

where $$0 \not> \mu \not> n-1.$$

Let $$m_1^{n-2} = \lambda + \lambda' n,$$

where $$1 < \lambda < n,$$

then $$(m_1+\kappa n)^{n-1} = m_1^{n-1} + m_1^{n-2} \cdot \kappa n(n-1) + \text{a multiple of } n^2$$
$$= 1 + (\lambda - \kappa\mu)n + \text{a multiple of } n^2.$$

Hence $(m_1+\kappa n)^{n-1}$ is divisible by n^2, if and only if $\lambda - \kappa\mu$ is divisible by n, *i.e.* if

$$\lambda\mu^{n-2} - \kappa\mu^{n-1} \text{ is divisible by } n,$$

or if

$$\lambda\mu^{n-2} - \kappa \text{ is divisible by } n.$$

Thus if κ is the remainder in dividing $\lambda\mu^{n-2}$ by n and $m_2 = m_1 + \kappa n$, then $(m_2+kn^2)^{n-1}$ is divisible by n^2, and there are therefore (excluding units) $n-2$ distinct values of m_2 corresponding to the different values of m_1.

It is obvious that if m_2 satisfy this condition, so do $m_2^2, m_2^3 \ldots m_2^{n-2}$. But if M_1 be a primitive root of n [*i.e.* if $n-1$ be the smallest value of m for which $M_1^m - 1$ is divisible by n], and M_2 the corresponding value of m_2, then $M_1, M_1^2, \ldots M_1^{n-2}$ on division by n must leave the remainders 2, 3, ... $(n-1)$ in some order: so too must $M_2, M_2^2, \ldots M_2^{n-2}$. Hence, we may take for our set of distinct solutions the remainders on dividing $M_2, M_2^2, \ldots M_2^{n-2}$ by n^2.

Just as m_2 (or M_2) was determined from m_1 (or M_1), so may m_3 (or M_3) be determined from m_2 (or M_2), and so on in succession. W. E. H.

254. [V. 1. a.] *"Approximately equal to."*

I should like to suggest the use of a broken sign of equality $\eqslantless$ for "approximately equal to."

Thus, I should write

$$\tan\theta = \tfrac{52}{1500} \eqslantless \tfrac{1}{30}, \text{ whence } \theta \eqslantless 2°.$$
$$x = \sqrt{2} - 1$$
$$\eqslantless .41.$$

The sign is a long and short $\eqslantless$, with a slight tick at the end of the short to avoid confusion with a hastily written $=$. C. S. J.

255. [K. 8.] Let A, B, C, D be the vertices of a quadrilateral; *EFG* its diagonal trjangle. L, L', M, M', N, N' the mid points of AC, BD, AB, DC, DA, CB respectively; *efg* the median triangle of the triangle *EFG*.

It is a defect in all our text-books to exhibit the collinearity of *LL'e* and to ignore the collinearities of *MM'g* and *NN'f*.

The four points A, B, C, D give rise to three quadrilaterals, the pairs of diagonals being (AC, BD); (AB, CD); (AD, BC).

Students are apt to think that LL' possesses some special property not shared by MM', NN'. W. Gallatly.

256. [K. 6. a; 13. c.] *On the area of a triangle, the equations of whose sides are given.*

Let the sides be

$$a_1x + b_1y + c_1 = 0, \quad \ldots\ldots(1)$$
$$a_2x + b_2y + c_2 = 0, \quad \ldots\ldots(2)$$
$$a_3x + b_3y + c_3 = 0, \quad \ldots\ldots(3)$$

and let (1) contain the vertices (x_2y_2), (x_3y_3), and similarly for the others.

Then the area $\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$;

$$\therefore\ \Delta\times\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} a_1x+b_1y_1+c, & 0, & 0 \\ 0, & a_2x_2+b_2y_2+c_2, & 0 \\ 0, & 0, & a_3x_3+b_3y_3+c_3 \end{vmatrix}.$$

Since $a_2x_1+b_2y_1+c_2=0$,

and $a_3x_1+b_3y_1+c_3=0$;

$$\therefore\ a_1x_1+b_1y_1+c_1=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\div\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix};$$

$$\therefore\ \Delta=\frac{1}{2}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2\cdot$$

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\times\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}\times\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

In the same way the volume of the tetrahedron given by the planes

$$a_1x+b_1y+c_1z+d_1=0, \quad\ldots\ldots(1)$$
$$a_2x+b_2y+c_2z+d_2=0, \quad\ldots\ldots(2)$$
$$a_3x+b_3y+c_3z+d_3=0, \quad\ldots\ldots(3)$$
$$a_4x+b_4y+c_4z+d_4=0, \quad\ldots\ldots(4)$$

is $$\frac{1}{3}\cdot\frac{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}}{D_1\,.\,D_2\,.\,D_3\,.\,D_4},$$

where D_1, D_2, D_3, D_4 are the minors of d_1, d_2, d_3, d_4 in the determinant in the numerator.

Let the plane (1) contain the vertices (x_2y_2), (x_3y_3), (x_4y_4), and similarly for (2), (3), (4).

Then the perpendiculars from (x_1y_1) on (1) is

$$(a_1x_1+b_1y_1+c_1z_1+d_1)\div\sqrt{a_1^2+b_1^2+c_1^2}$$

or $$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}\div D_1\sqrt{a_1^2+b_1^2+c_1^2}.$$

Therefore the area of the triangle formed by the intersections of the planes (2), (3), (4) with (1) is

$$\frac{1}{2}\cdot\frac{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}^2\times\sqrt{a_1^2+b_1^2+c_1^2}}{D_2\,.\,D_3\,.\,D_4}.$$

S. T. Shovelton.

257. [X. 4. b.] *On the graphical solution of equations.*

The following method is adapted to the solution of equations of the type

(α) $x^2\pm bx\pm c.$

(β) $\left.\begin{aligned} ax\pm by&=c, \\ xy&=d. \end{aligned}\right\}$

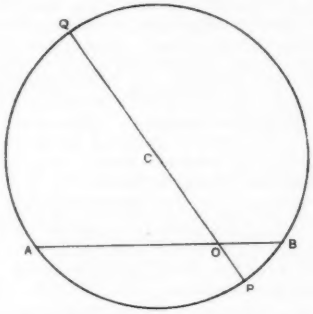
Take case (α).

Take a line AOB such that AO, OB are two factors of c. (These can always be found.)

Draw a circle through AB having radius $\frac{b}{2}$.

Let its centre be C.

Let a chord through C and O cut the circle in P, Q. OP, OQ represent the two solutions.



For $x^2 \pm bx - c$ (b and c being positive), the point O is taken externally.

The same method solves case (β).

AO, OB are two factors of abd.

The radius of the circle is $\frac{c}{2}$.

OP, OQ are ax, by respectively. Hence the solution.

NORMAN J. CLUGNELL.

QUERY.

(49) In Dr. Young's interesting and valuable paper (*Gazette*, pp. 155-6) he takes the view that an area involves the idea of an infinite sequence and a limit. Is this doctrine sound? When I say that a circle of radius r is equal in area to a quadrant of radius $2r$ an infinite sequence of dissections is of course implied. But if I say that the area of a circle is greater than that of the inscribed square, there is no such implication—the matter is tested at once without even one dissection. Is not the infinite sequence rather involved in the idea of equality? Inequality can be established in a finite number of steps. But equality has to be taken as the negation of inequality, and in many cases* brings in the idea of a limit of an infinite sequence when inequality does not: the two cases given are examples.

Dr. Young says it is easy to define the area of a triangle. But is it not practically an indefinable?

Does Dr. Young admit any distinction between the area itself and its numerical measure? Would it be right to say that the mistake against which he wishes to warn his readers is in assuming without proof that the area enclosed by any given contour is capable of accurate numerical measurement? This query applies to plane areas and volumes; curved lengths and surfaces must be taken as limits since they cannot be compared

*Perhaps in all; even when two equal straight lines are compared by superposition the judgment of the eye alone is not accepted, but an infinite series of imaginary microscopes of continually increasing power is called into play.

with straight lines and plane areas by superposition. May I suggest a further illustration of the idea of a limit in ordinary life in the phrase 'The train is moving'?

A. C. DIXON.

ANSWERS TO QUERIES.

[10, p. 95.] Prof. E. J. Nanson has referred (p. 132) to Cayley's published explanation; but a clearer statement was given by Cayley in a course of lectures on Differential Equations delivered in Cambridge in 1886. Just before leaving the subject of Singular Solutions, Cayley mentioned two cases in which irregular factors present themselves. The first, which he described as a *Reiterate Form*, occurs when for a certain value of c the equation $\phi(x, y, c)=0$ (from which the differential equation is derived) contains a *squared* factor: this factor will intrude itself when the envelope is derived from the differential equation. The second case is what here concerns us; this Professor Cayley explained as follows, illustrating it by instances in which $\phi(x, y, c)$ is quadratic in c:

The Coincident Consecutive. Suppose that

$$\phi(x, y, c) \equiv L(c+M)^2+N(Pc^2+2Qc+R), \quad \ldots\ldots(1)$$

the capital letters denoting functions of x and y. All the curves of the system touch $N=0$ at the points where it is met by $M+c=0$; and N proves to be a factor in the equation found by eliminating c from ϕ, and its differential coefficient with respect to c, viz.

$$(L+NP)(LM^2+NR)=(LM+NQ)^2. \quad \ldots\ldots(2)$$

But suppose M is a constant $=-k$; N is still a factor in (2), but is no longer touched by the curves (1); it is clear that, when $c=k$, $N=0$ is a part of the curve (1).

Approach the question differently, and consider the family of curves

$$L(c-k_1)(c-k_2)+N(Pc^2+2Qc+R)=0, \quad \ldots\ldots(3)$$

which for the two values k_1 and k_2 of c contain the curve $N=0$; the equation of their envelope is divisible by N except for the term

$$L^2(k_1-k_2)^2.$$

Thus, when two non-consecutive curves of the system have a common part, no extraneous-factor is introduced: when two *consecutive* curves of the family coincide (wholly or in part) the equation of the common part will be present in the result of finding the equation of the envelope in the usual manner.

H. W. RICHMOND.

[10, pp. 95, 132, 213.] In the various comments on the result of trying to find the envelope of the parabolas circumscribing the triangle ABC, by which we get the equation to the straight lines BC, CA, AB, no one seems to have noticed why these particular lines appear, and have a right to appear, in the result (rather than any other locus through A, B, C). *They are really limits of intersection of parabolas of the system whose parameters differ by indefinitely small quantities.*

Thus, the curves being (as given) $x^2-ax+2\lambda xy+\lambda^2(y^2-by)=0$, the form approximates to $x(x-a)=0$, when λ is a small quantity ϵ. The (true) intersections, other than ABC, of this particular curve with that of parameter θ are given by:

$$\theta+\epsilon=\frac{-2xy}{y^2-by}=\frac{2x}{b-y} \quad \ldots\ldots(1); \qquad \frac{\theta+\epsilon}{\theta\epsilon}=\frac{-2xy}{x^2-ax}=\frac{2y}{a-x} \quad \ldots\ldots(2);$$

whence $$y=\frac{-2a(\epsilon+\theta)}{(\epsilon-\theta)^2}=-\frac{a}{\epsilon}\left(\frac{2}{1-z}\right)\left(\frac{2}{1-z}-1\right), \text{ writing } z \text{ for } \frac{\theta}{\epsilon}.$$

Hence it readily appears that, as θ ranges from $-\epsilon$ to $+\epsilon$, y ranges from 0 to $\pm\infty$, according to the sign of ϵ; while (1) shows that x continues very

small, for finite values of y; (being smaller, the smaller the value of ϵ). Thus, *however small* δ *may be*, there are, within the extreme limits $\pm\delta$ for λ, curves whose intersections with each other approximate to the line $x=0$ *along its entire length*. Similarly for the other straight lines. They are a sort of condensed envelopes (if we might so extend the term) not of the system as a whole, but of extremely limited portions of the system; of which indeed they form a part—belonging to the three parabolas which reduce to parallel straight lines. PERCY J. HEAWOOD.

[28, p. 166.] Two solutions are given (by Mr. I. Arnold and Mr. C. Joss respectively) in *Educ. Times Reprint*, vol. 73, p. 42.

[34, p. 166.] Let D, E, F be the middle points of sides, D', E', F' the feet of perpendiculars, of the triangle ABC. Then, if K be the point of contact with BC of the circle escribed to it, we have

$$2KD=(s-b)-(s-c)=c-b\,;\ 2DD'=BD'-D'C=\frac{c^2-b^2}{a}. \quad \ldots\ldots\ldots(1)$$

Now if the nine-point circle be touched at X, between D and D', by a circle touching BC at K (which we know will be the escribed circle), KX produced will bisect the arc DD' (since the tangent at its middle point R is parallel to BC), and therefore will bisect the angle DXD' externally; so that we have: $DX : D'X :: KD : KD' :: a : a+b+c$ by (1).

Again, if P be a point on the nine-point circle between E and F, unit forces along PE, $PF \equiv$ force $2\cos\dfrac{B+C}{2}$ along PR, the bisector of the angle, $\equiv$ forces along PD, PD' each equal to

$$\frac{\cos\dfrac{B+C}{2}}{\cos\dfrac{B-C}{2}} \text{ (since angle } DPD' = \text{angle } DED' = C-B),$$

$$i.e.\ \text{each} = \frac{\sin(B+C)}{\sin B+\sin C} = \frac{a}{b+c}.$$

Hence unit forces along PD, PE, PF reduce to forces in PD, PD', which are to each other as $a+b+c : a$; their resultant therefore acts through the point X in the arc DD', determined before. (For PX makes PD, PD' angles equal to those at D', D of the triangle DXD'; whose sides are proportional to the two forces in question, and at the same inclination to each other.) Similarly with P in other positions, *or with the directions of some of the forces reversed*, the line of the resultant will pass through the other points of contact. PERCY J. HEAWOOD.

[34, p. 166.] The following solution has been obtained by Casey's method of using the circumcircle instead of the Nine-point circle, and producing the points required to double their distance from the orthocentre. The figure is readily drawn. If we take P any point on the circumcircle instead of on the N.P. circle, then the forces must act along PA', PB', PC', where $KA'=2KX$, etc., K being the orthocentre and X, Y, Z the mid points of the sides of $\triangle ABC$.

Again, if I be incentre and KI be produced to I' so that $II'=KI$, I' will be centre of a new circle touching circumcircle at a point T.

Suppose AI meets circumcircle in a; then a is middle point of arc BC. Again, if incircle originally touched BC at D and KD be produced to D' so that $KD=DD'$, aD' will meet circumcircle in the point T' where the circles centre I' and O touch (O is circumcentre).

[For if two circles touch, the line joining points of contact of parallel tangents passes through the point where they touch; and tangents at a and D' are obviously parallel.]

OI' also must pass through T.

Suppose AK meets circumcircle in λ and DC in L.

Then $KL = L\lambda$; also $KX = XA'$.

$\therefore$ $A'\lambda \parallel BC$; AOA' is a straight line, and A', B', C' the "extended" positions of the mid points of the sides lie on the circumcircle.

Consider forces R along $A'P$, PB', PC' respectively (R circumradius is taken for convenience).

Since Pa bisects $\angle B'PC'$ and since $\angle B'PC' = 180° - A$, resultant of R along PB', PC' is $2R\sin\frac{A}{2}$ along Pa.

Now $Oa = R$, $aI = aB = 2R\sin\frac{A}{2}$.

Also $\angle OaI = \angle OAI = \angle A'Pa$.

$\therefore$ OaI is Δ of forces and OI represents resultant of R along $A'P$ and $2R\sin\frac{A}{2}$ along Pa.

$\therefore$ resultant of the 3 forces is PT' (T' being on circumcircle), where PT' makes with PA' an angle $\pi - \angle aOI$.

Lastly, to show T and T' are the same points:

$$a\lambda = 2R\sin\frac{C-B}{2}, \text{ for } \angle aA\lambda = \frac{C-B}{2},$$

$$D'\lambda = 2DL = 2IA\sin\frac{C-B}{2}$$

and $\angle A'Aa = \angle D'\lambda a$;

$\therefore$ $\triangle$s $a\lambda D'$ and OIA are similar;

$\therefore$ $\angle AOI = \angle \lambda aD'$ or $\angle \lambda aT$, or for convenience arc λT.

Hence
$$\text{arc } A'T' = \pi - \angle aOI$$
$$= \pi - (\pi - 2 \text{ arc } A'a - \text{arc } \lambda T),$$

since $\angle A'Oa = 2\angle A'Aa$

and
$$\angle AOI = \angle \lambda aT$$
$$= \pi - (\pi - \text{arc } A'T), \text{ since arc } A'a = \text{arc } a\lambda,$$
$$= \text{arc } A'T;$$

$\therefore$ T and T' are the same points. Q.E.D.

If now everything be produced backwards to half its distance from the orthocentre we shall get proposition enunciated in the question.

If then we call the point where the incircle touches the N.P. circle T, and the points where the excircles touch it T_1, T_2, T_3, and if we consider the forces as positive if acting along PA', PB', or PC', and negative if acting along $A'P$, $B'P$, or $C'P$, it is obvious from the above that the resultant will pass through T if the forces have not all like signs. The resultant passes through T_1, T_2, or T_3 if the forces all have like signs, corresponding to the position of P in arcs AB, BC, and CA. But this requires proof.

W. F. BUSHELL.

[34, p. 166.] Using the figure, p. 86 of Richardson and Ramsey's *Modern Plane Geometry*, we have incentre I, circumcentre O. Join AI and produce to meet BC in R. Draw RS touching the incircle at S; from A', the mid-point of BC, draw $A'SS'$ to meet the incircle again at S', which is the point of contact of the incircle and nine-point circle, as proved in the above-named book.

Since the *equal* forces along PA', PB', PC' (the sense being adjusted) are to have a resultant along PS', then, taking moments about S', we have to prove that $\Sigma \sin S'PA' = 0$.

Now $\angle S'PA' = S'SR$, since the tangents at S and A' are parallel,

$$= \pi - A'SR.$$

Hence we must prove that $\Sigma \sin A'SR = 0$.

In the $\triangle A'SR$ we can easily shew that $SR = RX = (s-a)\dfrac{c-b}{c+b}$ and $A'R = \dfrac{a}{2} \cdot \dfrac{c-b}{c+b}$; also $\angle SRA' = C - B$;

$$\therefore\ A'S^2 = \frac{1}{4}\left(\frac{c-b}{c+b}\right)^2 \left\{a^2 + 4(s-a)^2 - 4a(s-a)\cos\overline{C-B}\right\},$$

and after reduction $A'S$ may be exhibited in the form $M(c-b)$, where

$$M = \frac{OI}{2R};$$

$$\therefore\ \sin A'SR = \frac{A'R}{A'S}\sin A'RS$$

$$\propto \frac{a}{c+b} \cdot \sin\overline{C-B}$$

$$\propto \frac{\sin\overline{C+B} \cdot \sin\overline{C-B}}{\sin C + \sin B}$$

$$\propto \sin C - \sin B.$$

Hence $\Sigma \sin A'SR = 0$. Q.E.D.

Similarly for the points of contact with the ex-circles.

Note.—It can be shewn that

$$\sin A'SR = \frac{R}{OI}(\sin C - \sin B) = \frac{c-b}{2OI} = \frac{A'X}{OI} = \sin OIX,$$

whence $A'SR = OIX$,

and $\cos A'SR = \dfrac{R}{OI}(1 - \cos B - \cos C)$.

Hence, the resultant of the three equal forces F (by resolving along PS')

$$= \Sigma F \cos A'SR = F \cdot \frac{R}{OI}\Sigma(1 - \cos B - \cos C)$$

$$= F \cdot \frac{R}{OI}(3 - 2\Sigma\cos A) = \frac{OI}{R} \cdot F.$$

Now, if three equal forces F act along BC, CA, AB, the resultant can easily be shewn to be $\dfrac{OI}{R} \cdot F$ acting $\perp$ to OI (see *Math. Gazette*, vol. ii. p. 191).

Hence the system through P is equivalent to this system as regards magnitude, in other words ABC is the Δ of forces, as may be otherwise seen, the directions being CB, BA, AC. We also note incidentally that $S'A' + S'B' + S'C' = 0$ algebraically, and that $S'D$ ($AD \perp$ to BC) is $\perp$ to the image of OI about BC.

F. GLANVILLE TAYLOR.

[38, p. 212.] In constructing a triangle when one of the data is the difference of base angles, the following properties are often useful:

Let the median AA' of the triangle meet the circumcircle in X, draw XX' parallel to CB to meet circumcircles again in X'.

Then

$$\angle AX'A' = B \sim C, \quad \ldots\ldots(1)$$

$$A'X = A'X', \quad \ldots\ldots(2)$$

$$AB \cdot AC = AA' \cdot AX'. \quad \ldots\ldots(3)$$

Proofs. In triangles $X'BA'$ and XCA', since XX' is parallel to CB, $CX = BX'$, $\angle XCA' = \angle X'BA'$, and $CA' = BA'$;

$\therefore \angle BX'A' = CXA' = CBA$, also $AX'B = ACB$;

$\therefore AX'A' = B - C$ and $AX' = A'X$.

Also CAA' and $X'AB$ are equiangular;

$\therefore AC : AA' = AX' : AB$; $\therefore AC \cdot AB = AA' \cdot AX'$.

In Q. 38, AA' is given and BC;

$\therefore$ since $AA' \cdot A'X = (\frac{1}{2}CB)^2$, $\therefore A'X$ is known.

On AA' describe a segment of a circle containing an angle $= B - C$. Inflect $A'X' = (CB)^2/4AA'$. Produce AA' to X so that $A'X = A'X'$, draw CB parallel to XX' and make

$$A'B = A'C = \frac{AB}{2}.$$

W. M. COATES.

[39, p. 212.] If OP, OQ are given tangents to a conic, at given points P, Q, and ON is a given normal, the pole of ON will lie on PQ (since ON passes through the pole of PQ), and is determined as the harmonic conjugate K' of the point K where ON meets PQ. If then $K'H$ is drawn perpendicular to the normal, it will be the corresponding tangent, and the conic is determined as touching three fixed straight lines at given points.

PERCY J. HEAWOOD.

[41, p. 212.] To construct a triangle ABC, given BC, the direction BX of BA, and CX the symmedian from C.

If CaB be any triangle, on the base BC, with its vertical angle $BaC = CBX$ (Ca meeting BX in A), its median Cf will be the symmedian of the triangle CBA, since the two triangles will be similar. Further, f, the middle point of Ba, will lie on a fixed circle which may be readily constructed: it will pass through B, through D the middle point of BC, and through N the foot of the perpendicular from C on BX, as well as through O, the centre of the fixed circle circumscribing BaC (which will be diametrically opposite to B).

We have then only to take f where CX meets the former circle and produce Bf to meet the latter in a, to secure that CX is the median of CaB, and the symmedian of CBA, where A is the intersection of BX with Ca.

There will be two solutions, one, or none, according as CX cuts, touches, or does not meet the smaller circle.

PERCY J. HEAWOOD.

[41, p. 212.] The late Col. Mannheim's solution is as follows: Let the symmedian from C cut AB in C'. Let CD be an arbitrary length on the $\perp$ from C, and let the circle on CD as diameter cut BC in B'. Through the image of B' in CC' draw a parallel to CC', cutting the circle in A, A''. Then CA', CA'' cut BC' in A, A_1, which are two positions of the required vertex.

For $CB' \cdot CB = CA' \cdot CA$ and the triangles ABC, $A'B'C$ are similar. The median CC' of $A'B'C$ is the homologue of the median CM of ABC. Hence the two medians are equally inclined to CA, CB; $\therefore CC'$ is a symmedian of ABC. Hence A is vertex required. Similarly for A_1.

[42, p. 212.] These points are discussed in the *Intermédiaire des Mathematiciens*, vol. vii., 1900, p. 389, by M. Paul Tannery and a contributor under the nom de guerre of Belga. For the words in question we must refer to the Greek terminology handed down to us by the Euclidean school. In the Tenth Book of Euclid's *Elements*, we find codified the theory of incommensurables created by Theaetetus, Plato's Athenian friend, and by Eudoxus of Cnidus. Now the ancients gave the name of *ῥηταί*, *i.e.* rationals, to the arbitrary lines (corresponding to our arithmetical units) with which they compared other lines, and the same term was applied to all lines commensurable in length with the former, *εὐθεῖα σύμμετρα μήκει*, or when squared, *εὐθεῖα δυνάμει σύμμετρα*. All other lines were called irrational, *ἄλογοι*. To every magnitude of the forms $a + \sqrt{b}$, $\sqrt{a} + \sqrt{b}$, $\sqrt[3]{a} + \sqrt[3]{b}$, and so on, they gave the name *εὐθεῖα ἐκ δύο ονομάτων*, line of two names; the

constituent parts being the terms, τά ὀνόματα. The first translators and commentators of the works of Euclid translated the Greek by *linea binominis*, *linea binomialis*, or simply *binomium*. The words are all found in low Latin, but there is no trace of a *binominalis*. Monomial and polynomial were constructed by analogy and are of course hybrids. In the 17th century *multinomium* in Latin and *multinome* in French were used more often than *polynome*. The question of the circumflex accent was decided in the negative by Bertrand, Secretary of the *Académie des Sciences*, who was also one of the Forty; but the Academy itself and Littré preferred to retain the accent. M. Tannery expresses the view that as binomial may be used adjectively there is no particular point in using either binomial or binominal. The latter form is the more regular, but both binomial and binominal may be defended.

[**43, p. 212.**] From the invaluable *Intermédiaire* we take the following. M. H. Brocard has found "momentum" (product of a force and the lever arm perpendicular to the direction of the force) in a letter from Wallis to Fermat, dated $\left.\begin{matrix}21\text{ Nov.}\\1\text{ Dec.}\end{matrix}\right\}1657$. Varignon, to whom is generally attributed the first use of the word, was then but a babe. To this M. Paul Tannery adds: "The Greeks called the tendency of the scale of the balance to descend ῥοπη, whence ἰσορροπία, equilibrium. This tendency is given as dependent on the weight and the arm of the lever in Aristotle, and in the Commentary of Eutocius on Archimedes' *Equilibrium of Planes*, though Archimedes had not yet arrived at the concept. Commandin, in 1506, translated the idea by the Latin *momentum* in this technical sense. Galileo extended the concept in his *Theory of Simple Machines*, published about 1593, in French by Mersenne in 1634. He later generalized it by a principle equivalent to that of virtual velocities, in his *Discorso intorno alle cose che stanno in sù l' acqua*, 1612—"a work the theoretic importance of which has been far too much overlooked."

EDITOR.

[**44, p. 212.**] Simon Stevinus used the "parallelogram of forces" occasionally, but preferred the "triangle of forces," as "the necessary terms are known to the triangle." H. BRAID, in the *Intermédiaire*.

[**46, p. 212.**] $e^{\pi\iota}=-1$ is well known,

it is $e^{-\frac{\pi}{2}}=\iota^{\iota}$.

$$\frac{\pi}{4}=\frac{e^{\pi}-1}{e^{\pi}+1}=\sum_{n=0}^{n=\infty}\frac{1}{4n^2+4n+2}$$ (Wolstenholme).*

$\frac{\pi^{\pi}e^{e}}{\pi^{e}e^{\pi}}$ is approximately unity (Barisien).

$e^{-\frac{\pi}{n}}+e^{-\frac{4\pi}{n}}+e^{-\frac{9\pi}{n}}+\ldots+e^{-(n-1)^2\frac{\pi}{n}}$ is almost an integer if n is an odd square (Glaisher, *Q.J.* xv. 125-127, 1878). M. H. BROCARD, *Intermédiaire*.

ERRATUM.

L.H. of 3rd equation in (6), p. 95; for a^2 read a^3.

NOTICES.

The Slide-Rule. R. G. BLAINE. Pp. xii, 152. 3rd Edition. (Spon.) 2s. 6d. The sub-title is "Some Quick and Easy Methods of calculating." The book gives an account of logarithms and their application to the ordinary slide-rule with the log log scale as well. Most of the ordinary rules are given, but

* *Problems*, 1664 (1), p. 289 (1891).

the most valuable mnemonic has not been given the prominence which experience shows it demands, viz., that in any setting if a and b on the rule are opposite x and y on the slide respectively, then $\frac{a}{x,\ \sin x,\text{ or }\tan x} = \frac{b}{y,\ \sin y,\text{ or }\tan y}$ according to the scales used. The value of the mnemonic is that the positions of the figures on the slide-rule correspond exactly to those in the formulae, so that given any three, the fourth quantity of the ratios is easily found. Beginners find this rule extremely easy to remember. The author makes general use of Dally's rule for determining the decimal place. There are 137 examples on engineering, mechanical and electrical formulae, many of them worked out and answers given to all. The book should be very useful to the beginner. W. M. ROBERTS.

Text Book of Mechanics. Vol. 2. **Kinematics and Kinetics.** By LOUIS A. MARTIN, Jr. (Wiley & Sons.)

The first volume on Statics has already been noticed in the *Gazette*. This second volume is intended for students who possess a little knowledge of the calculus.

It maintains the good qualities which characterised the first volume. The attention paid to angular velocity and rotation about fixed axes are features of interest. C. S. J.

Brief Biographical Sketch of Robert Rawson. By R. HARLEY, F.R.S. (J. Clark & Co., Oxford.)

Those who know something of the author's powers as a *raconteur* and have been privileged to hear from his lips anecdotes and reminiscences of the contemporaries he has outlived, will probably hasten to get this booklet and make themselves acquainted with its contents. Rawson's career was remarkable. It recalls in its broad outlines that of the dalesman John Dawson,* but the difficulties he surmounted seem greater even than those encountered by the Yorkshireman. His love of mathematical studies, the peculiar train of circumstances which released him from the toil of a miner's life, his career as a teacher and an investigator are here well told. It is profitable for teachers to read now and then the history of the self-taught. In particular, the power of the *puzzle* in stimulating hard work is worthy of notice. For Rawson was started by the arithmetical conundrums given for solution in a provincial newspaper on a course which took him ultimately as far as the determination of the 'complete cubic differential resolvent' printed *in extenso* in the *British Association Report for 1886*, pp. 439-443. E. M. LANGLEY.

BOOKS, ETC., RECEIVED.

American Journal of Mathematics. Vol. XXX. No. 1. Edited by F. MORLEY. $1. 50. Jan. 1908. (Kegan Paul.)

Plane Quintic Curves which possess a Group of Linear Transformations. V. SNYDER. *On Birational Transformations of Curves of High Genus.* V. SNYDER. *Surfaces with the same Spherical Representation of their Lines of Curvature as Spherical Surfaces.* L. P. EISENHART. *On the Factoring of Composite Hypercomplex Number Systems.* H. B. LEONARD. *A New Method in Geometry.* E. LASKER. *Groups generated by n Operators, each of which is the product of the* $n-1$ *remaining ones.* G. A. MILLER.

A Treatise on the Integral Calculus, founded on the Method of Rates. By W. W. JOHNSON. First Edition. xiv+440 pp. $3. 1907. (Wiley, Chapman & Hall.)

Reference Book for Statical Calculations. By F. RUFF. Vol. I. (Rapid Statics, Force Diagrams for Frameworks, etc.) 8s. 1905. (Spon.)

A First Course in the Differential and Integral Calculus. By W. F. OSGOOD. Pp. xv, 423. 10s. 6d. 1907. (Macmillan Co.)

Introduction to Higher Algebra. By MAXIME BÔCHER and E. DUVAL. Pp. xii, 321. $1. 90. 1907. (Macmillan Co.)

*Famous as having been, in the seclusion of a Yorkshire village, the private tutor of eight Senior Wranglers.

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